

Volume independence in large N_c QCD-like gauge theories

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ABSTRACT: Volume independence in large N_c gauge theories may be viewed as a generalized orbifold equivalence. The reduction to zero volume (or Eguchi-Kawai reduction) is a special case of this equivalence. So is temperature independence in confining phases. A natural generalization concerns volume independence in “theory space” of quiver gauge theories. In pure Yang-Mills theory, the failure of volume independence for sufficiently small volumes (at weak coupling) due to spontaneous breaking of center symmetry, together with its validity above a critical size, nicely illustrate the symmetry realization conditions which are both necessary and sufficient for large N_c orbifold equivalence. The existence of a minimal size below which volume independence fails also applies to Yang-Mills theory with antisymmetric representation fermions [QCD(AS)]. However, in Yang-Mills theory with adjoint representation fermions [QCD(Adj)], endowed with periodic boundary conditions, volume independence remains valid down to arbitrarily small size. In sufficiently large volumes, QCD(Adj) and QCD(AS) have a large N_c “orientifold” equivalence, provided charge conjugation symmetry is unbroken in the latter theory. Therefore, via a combined orbifold-orientifold mapping, a well-defined large N_c equivalence exists between QCD(AS) in large, or infinite, volume and QCD(Adj) in arbitrarily small volume. Since asymptotically free gauge theories, such as QCD(Adj), are much easier to study (analytically or numerically) in small volume, this equivalence should allow greater understanding of large N_c QCD in infinite volume.

KEYWORDS: $1/N$ Expansion, Lattice Gauge Field Theories.

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Contents

1. Introduction and Summary	1
2. Spacetime volume independence	4
2.1 Volume reduction as an orbifold projection	4
2.2 Volume expansion as an orbifold projection	7
2.2.1 Adjoint representation matter fields	9
2.2.2 Fundamental representation matter fields	9
2.2.3 Tensor representation matter fields	10
2.3 Large N_c equivalence	10
2.4 Temperature dependence	17
3. Eguchi-Kawai reduction for QCD	18
4. Theory space volume independence	22
4.1 Volume reducing projections	23
4.2 Volume enlarging projections	24
4.3 Large N equivalence	25
5. Discussion	25
A. More on volume expansion as an orbifold projection	27

1. Introduction and Summary

Back in 1982, Eguchi and Kawai [1] argued that properties of $U(N_c)$ Yang-Mill theory, formulated on a periodic lattice, are independent of the lattice size in the $N_c \rightarrow \infty$ limit. It was quickly understood that this large N_c equivalence (relating a pair of theories in different volumes) is valid only when certain global symmetries are not spontaneously broken [2, 3], and applies only to appropriate observables. When the theory is defined, for simplicity, in a d -dimensional periodic hypercube of size L , recent work has shown that large N_c volume independence is valid (in $d > 2$) only above a critical size, $L > L_c(d)$. The critical size $L_c(d)$ depends on the lattice spacing, in addition to dimension, but approaches a finite physical value in the continuum limit [4, 5].

Large- N_c equivalences between pairs of theories related by so-called “orbifold” projections have also received considerable attention in recent years. (See, for example, Refs. [6–13].) In this context, orbifold projection is a technique for constructing a “daughter” theory, starting from some “parent” theory, by retaining only those fields which are invariant under a discrete symmetry group of the parent theory. For orbifold projections based on cyclic groups, Refs. [6, 7] proved necessary and sufficient conditions for equivalence of parent and daughter theories in the large N_c limit, and clarified previous confusion in the literature regarding the appropriate mapping of information between theories. In complete analogy with volume independence, large N_c orbifold equivalence is applicable only when certain global symmetries are not spontaneously broken, and applies only to a restricted class of observables.

In this paper, we discuss the relation between volume independence and non-perturbative orbifold equivalence in the large N_c limit. The mapping which takes a $U(N_c)$ gauge theory in volume L^d to the same theory in a smaller volume $(L')^d$, with L/L' any integer, may be viewed as a generalized orbifold projection which eliminates degrees of freedom that are not invariant under a discrete translation group.¹ We show that inverse mappings, which blow up a smaller volume theory into a larger volume, may also be understood as orbifold projections, specifically ones which eliminate degrees of freedom that are not invariant under a discrete Abelian subgroup of the gauge (and flavor) symmetry group. Interpreting these volume-changing mappings as orbifold projections applies not only to pure Yang-Mills theories, but also to theories with fundamental, adjoint, or rank-two tensor representation matter fields.²

Viewing volume changing mappings between gauge theories as generalized orbifold projections allows a simple proof of large N_c equivalence using the methods of Refs. [6, 7], and also clarifies how the large N_c equivalence applies to connected correlators (in addition to expectation values) of operators in the appropriate “neutral” sectors of either theory. The validity of large N_c orbifold equivalence depends on certain symmetry realization conditions — the discrete symmetries used in either direction of these projections must not be spontaneously broken. These symmetry realization conditions are both necessary and sufficient [7]. As we discuss, the numerical results of Ref. [5] are completely in accord with, and nicely illustrate, these general conditions.

A special case of large N_c volume independence, which we discuss, is large N_c temperature independence. For mappings which change the size of a single periodic dimension, interpreted as the Euclidean time direction, the symmetry realization condition necessary for large N_c equivalence is just the condition that the theory be in a confining phase. Large N_c temperature independence implies that the leading large N_c behavior of expectation values and connected correlators of suitable operators are temperature independent in any confining phase.

In contrast to other examples of large N_c orbifold (or “orientifold”) equivalences, such as those discussed in Refs. [8–11], large N_c volume (or temperature) independence cannot be

¹In this paper, we are only concerned with theories formulated on toriodal compactifications of flat space.

²For earlier, somewhat related work, see Refs. [14, 15].

seen in a perturbative analysis — it is intrinsically non-perturbative. As we will discuss, this reflects the fact that unbroken center symmetry is a necessary condition for large N_c volume independence, and perturbative expansions do not respect the center symmetry.

For later convenience, let $\text{QCD}(\text{Adj})$ denote a $U(N_c)$ gauge theory with one or more massless adjoint representation Majorana fermions, and let $\text{QCD}(\text{AS})$ [or $\text{QCD}(\text{S})$] denote a $U(N_c)$ gauge theory with one or more massless Dirac fermions in the rank-two antisymmetric [or symmetric] tensor representation.³ Single-flavor massless $\text{QCD}(\text{Adj})$ is precisely $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. And at $N_c = 3$, $\text{QCD}(\text{AS})$ is the same as ordinary QCD with fundamental representation fermions. Hence, $\text{QCD}(\text{AS})$ is a generalization of ordinary QCD to arbitrary N_c in which a single flavor of fermions will continue to play a significant role even at $N_c = \infty$ [10, 11].

In pure $U(N_c)$ Yang-Mills theory, as noted above, volume independence fails for sufficiently small volumes due to a change in symmetry realization. But, as we discuss, the addition of massless adjoint representation fermions (with periodic boundary conditions) prevents this phase transition. Consequently volume independence in $\text{QCD}(\text{Adj})$ remains valid down to arbitrarily small volumes.

This volume independence of $\text{QCD}(\text{Adj})$ has interesting implications for large N_c QCD, because a separate large- N_c equivalence [10, 16] relates $\text{QCD}(\text{Adj})$ and $\text{QCD}(\text{AS})$, provided the latter theory does not spontaneously break charge conjugation symmetry [17]. In sufficiently small volumes (with periodic boundary conditions for the fermions) $\text{QCD}(\text{AS})$ *does* undergo charge conjugation symmetry breaking, thus invalidating the large- N_c equivalence to $\text{QCD}(\text{Adj})$ [17]. However, all available evidence is consistent with $\text{QCD}(\text{AS})$ having unbroken charge conjugation symmetry in sufficiently large volume [18]. Provided this is true, combining the large- N_c equivalence between $\text{QCD}(\text{AS})$ and $\text{QCD}(\text{Adj})$, in sufficiently large volume, with the large- N_c volume independence of $\text{QCD}(\text{Adj})$, allows one to relate properties of $\text{QCD}(\text{AS})$ in large (or infinite) volume to corresponding quantities in $\text{QCD}(\text{Adj})$ in arbitrarily small volume. As we will discuss, this equivalence applies to both expectation values and connected correlation functions of suitable (charge conjugation even) bosonic operators. In the zero volume limit, this shows that a simple matrix model (or “reduced” theory) can reproduce the leading large N_c behavior of a large class correlation functions in infinite volume $\text{QCD}(\text{AS})$ — which is a natural generalization of real QCD to large N_c .⁴

Following this extended discussion of volume independence in spacetime, we show that a completely analogous approach demonstrates “theory space volume independence” in quiver

³We will also use $\text{QCD}(\text{AS/S})$ to denote either $\text{QCD}(\text{AS})$ or $\text{QCD}(\text{S})$ in contexts where the same results apply to either theory.

⁴This formulation of a reduced matrix model for QCD, in which fermions prevent the spontaneous breaking of center symmetry, is complementary to previous approaches yielding “quenched” [2] or “twisted” [19, 20] reduced models, where modifications to the action are introduced by hand in order to prevent unwanted symmetry breaking. (See Ref. [21] for a good pedagogical discussion of quenched and twisted reduced models.) In particular, these approaches lack direct applicability to connected correlators.

gauge theories. These are theories with product gauge groups and bifundamental matter fields. “Theory space” is a convenient graphical representation of the gauge group and matter field content of such theories [22,23]. We show that one may reduce a quiver gauge theory with multiple equivalent gauge group factors to another quiver theory with a smaller product gauge group by performing a projection based on a subgroup of the symmetry group permuting equivalent gauge group factors. Alternatively, a quiver gauge theory can be “blown up” to produce bigger quiver theories with enlarged product gauge groups by performing orbifold projections based on a subgroup involving the global flavor symmetry. Projections which increase the volume (in either spacetime or theory space) act as inverses of the projections which reduce volume.⁵ In all cases, the condition that the relevant discrete symmetries used in the projections not be spontaneously broken is both necessary and sufficient for large- N_c equivalence between pairs of quiver gauge theories related by these orbifold projections.

It should be noted that generic orbifold projections do not produce examples of large N_c equivalence (even when one requires that neither the parent nor daughter theories break any symmetries spontaneously).⁶ Precise conditions which delineate which generalized orbifold projections do, or do not, lead to examples of large N_c equivalence are not currently known. However, our results support a simple and natural conjecture:

Two theories, A and B , each with smooth large N_c limits, will have coinciding large N_c limits, within appropriate neutral sectors, if a generalized orbifold projection exists which maps theory A with $N_c = kN$ to theory B with $N_c = N$, for some k , and an “inverse” orbifold projection exists which maps theory B with $N_c = mN$ back to theory A with $N_c = N$, for some m .

In addition to the examples discussed in this paper, and their obvious generalizations, this conjecture is also supported by a wide variety of examples involving mappings between unitary, orthogonal, and symplectic gauge groups. This is discussed in a forthcoming paper [24].

2. Spacetime volume independence

2.1 Volume reduction as an orbifold projection

Consider a d -dimensional $U(N_c)$ gauge theory, with or without matter fields, formulated in a finite spacetime volume Λ taken, for simplicity, to be a periodic hypercube of volume L^d .⁷

⁵This, of course, is somewhat sloppy terminology since both projections reduce the number of degrees of freedom in a theory. The net effect of a projection times its “inverse” is a reduction of N_c by a fixed multiplicative factor, while leaving the form of the theory otherwise unchanged.

⁶For example, a \mathbb{Z}_2 orbifold projection can take a $U((k+l)N)$ theory with adjoint matter to a $U(kN) \times U(lN)$ theory with bifundamental matter. But, unless $k = l$, there is no simple mapping which relates expectation values or correlators of observables of these theories in the $N \rightarrow \infty$ limit.

⁷Generalizing this discussion to the case of anisotropic volumes, or mappings which shrink different dimensions by different amounts, is completely straightforward. For ease of notation, we will stick to simple cubic volumes in our discussion.

Because of the periodic boundary conditions, the translation invariance of the theory is a compact $U(1)^d$ global symmetry. Choose any integer $K > 1$, and consider the discrete $(\mathbb{Z}_K)^d$ subgroup of the translation symmetry generated by translations through multiples of the distance $L' \equiv L/K$ along any coordinate direction. Let \mathcal{T}_K denote this discrete translation group, and let Λ' denote a periodic hypercube of volume $(L')^d$.

The reduction of the theory from the volume Λ to the smaller volume Λ' may naturally be viewed as an orbifold projection in which one eliminates from the theory all degrees of freedom which are not invariant under the translation subgroup \mathcal{T}_K . For the criterion of ‘non-invariance’ to be meaningful, one must first choose variables which transform irreducibly under the symmetry subgroup defining the projection. In this case, that just means Fourier transforming fields from spacetime to momentum space. In the original volume Λ , momenta are quantized in multiples of $2\pi/L$, so for some generic field Φ ,

$$\Phi(x) = \sum_{n \in \mathbb{Z}^d} \Phi_n e^{2\pi i n \cdot x / L}, \quad (2.1)$$

where n is an integer-valued d -component vector.⁸ Eliminating degrees of freedom which are not invariant under the translation subgroup \mathcal{T}_K simply means setting to zero all Fourier coefficients Φ_n for which the components of n are not integer multiples of K . This is the same as averaging $\Phi(x)$ over all translations along coordinate axes by multiples of L' . Consequently, the projection takes

$$\Phi(x) \longrightarrow \tilde{\Phi}(x) \equiv \sum_{m \in \mathbb{Z}^d} \tilde{\Phi}_m e^{2\pi i m \cdot x / L'}, \quad (2.2)$$

where $\tilde{\Phi}_m = \Phi_{mK}$. This is the Fourier series for a field in the smaller periodic volume Λ' . The net effect of this projection is to replace every field, which was initially periodic with period L , by a projected field which is periodic with period L' . In other words, the projection is the same as imposing, on every field of the theory, a constraint of shorter periodicity,

$$\Phi(x + L' \hat{e}_\nu) = \Phi(x), \quad (2.3)$$

where $\{\hat{e}_\nu\}$ denote coordinate basis vectors. We have belabored the discussion of this simple projection in order to emphasize that “projecting out” degrees of freedom which are not invariant under some symmetry subgroup is exactly equivalent to imposing a set of constraints, such as Eq. (2.3), expressing invariance of fields under the chosen symmetry transformations. This is equally true for all other orbifold projections.

⁸For gauge fields, we assume a choice of gauge which preserves periodic boundary conditions. Or one may consider a lattice gauge theory without gauge fixing, in which case L (divided by the lattice spacing, assumed to be set to one) must be multiple of K , and (2.1) will have a cutoff of L on the components of n , for both link and site variables. If any fields (such as fermions) are defined with antiperiodic boundary conditions, then their allowed momenta in volume Λ will have components of the form $(2n+1)\pi/L$. In this case, K must be odd so that the allowed momenta in the smaller volume Λ' are commensurate with allowed momenta in the larger volume. Although Eqs. (2.1)–(2.3) show the case of periodic fields, their extension to antiperiodic fields is obvious.

The projection (2.2), applied to all fields of the theory, may be regarded as defining a mapping between observables of the “parent” theory, defined in the volume Λ , and the “daughter” theory, defined in Λ' . Under this mapping, all observables which carry momenta that are not quantized in units of $2\pi/L'$ map to zero.

In theories without fundamental representation matter fields (but possibly with adjoint or rank-two tensor representation fields) the natural gauge invariant observables are “single-trace” operators which are Wilson loops, or Wilson loops decorated with arbitrary insertions of matter fields along the loop. Such single-trace observables may be classified by their net winding numbers around each cycle of the periodic volume in which the theory is defined.⁹ The mapping (2.2) defines a one-to-one correspondence between gauge invariant single-trace observables in the parent theory which are invariant under \mathcal{T}_K (*i.e.*, observables averaged over all translations by multiples of L') and single-trace observables in the daughter theory whose winding numbers are integer multiples of K . If fundamental representation fields are present, then the mapping gives a one-to-one correspondence between mesonic observables in the parent which are invariant under \mathcal{T}_K and arbitrary mesonic observables in the daughter theory. These classes of gauge-invariant observables, in both parent and daughter, will be termed “neutral”.

Since the action of the parent theory contains a spacetime integral over the volume Λ , which is K^d times the volume of Λ' , the projection (2.2) does not map the action of the parent theory to the action of the daughter theory. Rather,

$$S_{\text{parent}} \longrightarrow K^d S_{\text{daughter}} . \quad (2.4)$$

The prefactor is the ratio of the number of degrees of freedom in parent and daughter theories. This form of the relation between parent and daughter actions, involving a rescaling by an overall multiplicative factor, is common to all orbifold projections but has, on occasion, been misunderstood (as noted in Ref. [12]).

For a lattice gauge theory (on a simple cubic lattice, with the lattice spacing set to one), if one chooses $K = L$ then this volume-reducing projection maps the parent theory defined on an L^d site lattice to a daughter theory defined on a lattice with just one site. For the specific case of pure Yang-Mills theory with the usual Wilson action,

$$S_W \equiv \frac{N_c}{\lambda} \sum_{x \in (\mathbb{Z}_L)^d} \sum_{\mu < \nu} \text{tr} \left(U_\mu[x] U_\nu[x + \hat{e}_\mu] U_\mu^\dagger[x + \hat{e}_\nu] U_\nu^\dagger[x] + \text{h.c.} \right) , \quad (2.5)$$

⁹In $U(N)$ theories with only adjoint matter fields, winding numbers are signed integers. In theories with rank-two symmetric or antisymmetric tensor representation matter, one should regard winding numbers as unsigned integers because gauge invariant single-trace observables such as $\text{tr}(U[C_{x \rightarrow y}] \Phi_y U[C'_{y \rightarrow x}]^* \Phi_x^*)$ cannot be assigned a consistent orientation. ($U[C_{x \rightarrow y}]$ denotes a parallel transporter along a curve C running from point x to y .)

the result is the Eguchi-Kawai model, which is a $U(N_c)$ matrix model with d independent unitary matrices $\{U_\nu\}$ and action

$$S_{\text{EK}} \equiv \frac{N_c}{\lambda} \sum_{\mu < \nu} \text{tr} (U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \text{h.c.}) . \quad (2.6)$$

Here $\lambda \equiv g^2 N_c$ is the (bare) 't Hooft coupling, which is held fixed as $N_c \rightarrow \infty$. (The conventional lattice coupling $\beta \equiv 2N_c^2/\lambda$.)

2.2 Volume expansion as an orbifold projection

To represent a volume-increasing mapping as an orbifold projection, it is easiest to use the language of lattice gauge theory.¹⁰ Before considering more general examples, it is useful to start with the simplest case of the single-site $U(N_c)$ Eguchi-Kawai model. (Volume-enlarging projections starting from an arbitrary size lattice are discussed in the Appendix.) The addition of matter fields will be considered below.

Choose an integer $K > 1$ and let N_c be divisible by K^d , so that $N_c = K^d N$ for some integer N . We wish to show that a $(\mathbb{Z}_K)^d$ orbifold projection can map this theory into a d -dimensional $U(N)$ pure gauge theory on a periodic K^d size lattice.¹¹

The Eguchi-Kawai (EK) model is invariant under $U(N_c)$ transformations which act by conjugation,

$$U_\mu \rightarrow V U_\mu V^\dagger, \quad V \in U(N_c). \quad (2.7)$$

This is the reduction to one site of the usual action of gauge transformations on link variables, $U_\mu[x] \rightarrow V[x] U_\mu[x] V[x+\hat{e}_\mu]^\dagger$. The EK model is also invariant under a $U(1)^d$ symmetry which multiplies each matrix by an arbitrary phase,

$$U_\mu \rightarrow e^{i\alpha_\mu} U_\mu. \quad (2.8)$$

This is the reduction to one site of what is commonly called center symmetry, which is the invariance of gauge theories with periodic boundary conditions (and only adjoint representation matter fields) under gauge transformations which are not themselves periodic, but are periodic up to some element of the center of the gauge group [26, 27].

The required projection will select degrees of freedom that are invariant under a $(\mathbb{Z}_K)^d$ symmetry subgroup which is embedded non-trivially within the full $U(1)^d \times U(N_c)$ symmetry group. The net effect of the projection is to impose a set of constraints having the form

$$U_\mu = \begin{cases} \gamma_\nu U_\mu \gamma_\nu^\dagger e^{2\pi i/K}, & \mu = \nu; \\ \gamma_\nu U_\mu \gamma_\nu^\dagger, & \mu \neq \nu, \end{cases} \quad (2.9)$$

¹⁰As discussed below, volume enlarging projections involve the center symmetry of the theory in an essential fashion. In a lattice formulation of the theory (on a cubic lattice) this invariance conveniently appears as a simple global symmetry corresponding to phase rotations of all link variables pointing in a given direction.

¹¹Generalizing the following treatment to the case of a $\mathbb{Z}_{K_1} \times \mathbb{Z}_{K_2} \times \cdots \times \mathbb{Z}_{K_d}$ projection mapping the EK model to a pure gauge theory on a periodic lattice of size $K_1 \times K_2 \times \cdots \times K_d$ is straightforward.

for all μ, ν , with $\{\gamma_\nu\}$ a particular set of d mutually commuting $(K^d N) \times (K^d N)$ unitary matrices whose eigenvalues are all K -th roots of unity, each with multiplicity $K^{d-1}N$. An explicit definition of these matrices is given in the Appendix.

If one block-decomposes the matrix U_μ into $K^d \times K^d$ blocks, each of which is $N \times N$, then the constraint (2.9) eliminates all but K^d of these blocks. The unitarity condition for the full matrix, $U_\mu U_\mu^\dagger = 1$, implies that each one of the surviving blocks must be an $N \times N$ unitary matrix. The constraint reduces the $U(N_c)$ symmetry (2.7) of the EK model to the smaller product group $[U(N)]^{K^d}$ — which is the full gauge group for a $U(N)$ lattice gauge theory on a K^d site lattice. Each surviving block of U_μ may be naturally associated¹² with an individual link pointing in the μ direction of a periodic cubic lattice Λ of size K^d , in such a fashion that the EK model action (2.6), evaluated with matrices satisfying the constraints (2.9), becomes precisely the Wilson action (2.5) for the lattice Λ and gauge group $U(N)$. The product of $U(N_c)$ Haar measures of the EK model, $\prod_\mu dU_\mu$, similarly reduces to the the product of Haar measures, $\prod_{x,\mu} dU_\mu[x]$, appropriate for a $U(N)$ gauge theory on Λ .

Under this mapping, every single-trace observable of the EK model whose winding numbers are integer multiples of K (*i.e.*, where the net number of U_μ minus U_μ^\dagger matrices, for each μ , are zero mod K) is mapped to a Wilson loop on Λ averaged over all translations,

$$\frac{1}{N_c} \text{tr} (U_{\mu_1} U_{\mu_2} U_{\mu_3} \cdots) \longrightarrow \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \frac{1}{N} \text{tr} (U_{\mu_1}[x] U_{\mu_2}[x+\hat{\mu}_1] U_{\mu_3}[x+\hat{\mu}_1+\hat{\mu}_2] \cdots) , \quad (2.10)$$

where $|\Lambda| = K^d$ is the number of sites of the lattice Λ . And every single-trace observable with non-zero winding (mod K) maps to zero under the projection (2.9).

Because of the overall factor of the rank of the gauge group in the actions (2.5) and (2.6), this projection from the $U(K^d N)$ EK model to a $U(N)$ gauge theory on lattice Λ maps the EK action to K^d times the Wilson action on Λ , so once again

$$S_{\text{parent}} \longrightarrow K^d S_{\text{daughter}} , \quad (2.11)$$

(with the actions on both sides defined with coinciding values of the 't Hooft coupling).

As discussed in the Appendix, this volume-enlarging orbifold projection may be generalized to the case of an initial periodic lattice Λ' of arbitrary size $(L)^d$ and a final periodic lattice Λ of size $(KL)^d$, for any integer $K > 1$. Therefore, it is completely natural to view these volume-enlarging orbifold projections as the “inverses” of the volume-reducing projections discussed in the previous subsection.

¹²The appropriate association of surviving blocks in U_μ with links in the lattice Λ is spelled out in detail (and in more generality) in the Appendix. But the basic idea is simple: Arbitrarily pick a surviving block of U_1 and some site x of the lattice Λ , and map the chosen block of U_1 to the link extending from site x in the x -direction. The term $\text{tr}(U_1 U_2 U_1^\dagger U_2^\dagger)$ in the EK model action (2.6) couples the chosen block of U_1 to precisely one surviving block of U_2 (and one block of U_1^\dagger and one block of U_2^\dagger). Map these blocks to the links forming the plaquette in the x - y plane which starts at site x . Continuing in the same fashion, considering plaquettes containing one or more already-assigned links, leads to a one-to-one mapping from surviving blocks of the matrices $\{U_\mu\}$ to links of the lattice Λ .

2.2.1 Adjoint representation matter fields

Generalizing these volume-enlarging projections to theories with adjoint representation matter fields (which may be either fermions or scalars) is straightforward. Consider, for example, adding an additional Hermitian matrix Φ to the one-site EK model, with action

$$S_{\text{parent}} = S_{\text{EK}} + N_c \text{tr} \left[\kappa \sum_{\mu} \Phi U_{\mu} \Phi U_{\mu}^{\dagger} + V(\Phi) \right]. \quad (2.12)$$

The hopping parameter κ and the scalar potential $V(\Phi)$ (which is some univariate function, bounded from below) are to be held fixed as $N_c \rightarrow \infty$. Note that the $U(1)^d$ transformations (2.8) remain a symmetry in the presence of adjoint representation matter fields. The specific $(\mathbb{Z}_K)^d$ symmetry transformations which define the projection of interest act on Φ by conjugation, so the constraints (2.9) on $\{U_{\mu}\}$ are simply augmented by the conditions

$$\Phi = \gamma_{\nu} \Phi \gamma_{\nu}^{\dagger}, \quad \nu = 1, \dots, d, \quad (2.13)$$

for the matrix Φ . The net effect of the constraint (2.13) is to reduce Φ to block diagonal form, with K^d blocks, each of which is $N \times N$. Each one of these blocks may be associated with a site of the lattice Λ in such a way that the result is a conventional $U(N)$ lattice gauge theory with an adjoint representation Hermitian scalar field. More precisely, the action (2.12) of the parent theory is mapped to K^d times the daughter theory action [*cf.* (2.11)], with

$$S_{\text{daughter}} = S_W + N \sum_{x \in \Lambda} \text{tr} \left\{ \kappa \sum_{\mu} \Phi[x] U_{\mu}[x] \Phi[x + \hat{\mu}] U_{\mu}^{\dagger}[x] + V(\Phi[x]) \right\}. \quad (2.14)$$

As in the pure gauge case, there is a one-to-one mapping between single trace observables in the parent matrix model whose winding numbers are zero modulo K , and single trace observables in the daughter lattice theory which are averaged over lattice translations.¹³

2.2.2 Fundamental representation matter fields

Defining an analogous volume-enlarging projection for theories with fundamental representation matter fields is, at first sight, problematic since fundamental representation fields break the $U(1)^d$ center symmetry which was an essential ingredient in the projection (2.9). However, this is not an insurmountable obstacle for theories with a $U(N_f)$ flavor symmetry. One may choose to regard a theory with a global flavor symmetry as the zero-coupling limit of a theory in which the $U(N_f)$ flavor symmetry is weakly gauged. In other words, a theory with a

¹³If the action of the starting theory contains multi-trace operators, then the direct application of a volume-enlarging projection leads to a non-local daughter theory whose action contains terms involving multiple integrals (or lattice sums) over spacetime. However, such non-local terms may be eliminated by adding to the action terms of the form $\sum_{x,y} [\mathcal{O}(x) - \mathcal{O}(y)]^2$ involving the zero momentum variance (or higher cumulants) of gauge invariant operators. Adding such terms does not affect the leading large N_c dynamics, within the neutral sector, provided translation invariance is not spontaneously broken.

$U(N_c)$ gauge group and fundamental representation matter with $U(N_f)$ flavor symmetry may equally well be regarded as a $U(N_c) \times U(N_f)$ gauge theory with bifundamental matter, in the limit of vanishing coupling for the $U(N_f)$ gauge group. The resulting $U(N_c) \times U(N_f)$ theory is precisely a quiver gauge theory of the class discussed in section 4. This reinterpretation is helpful because the $U(N_c) \times U(N_f)$ theory, with bifundamental matter, now contains a $U(1)^d$ center symmetry (which transforms the $U(N_c)$ and $U(N_f)$ gauge fields just like the pure Yang-Mills case, and leaves the matter fields invariant). If N_f is divisible by K^d , so that $N_f = K^d n_f$, then a volume-enlarging projection may be defined by suitably embedding the $(\mathbb{Z}_K)^d$ symmetry determining the projection into the product of the gauge and center symmetry groups of the $U(N_c) \times U(N_f)$ quiver theory. The resulting constraints have the effect of reducing the matter fields, viewed as an $N_c \times N_f$ matrix, to a block-diagonal form with K^d blocks each of which is $N \times n_f$. If N_f/N_c is held fixed as $N_c \rightarrow \infty$, then this volume-enlarging projection may again naturally be viewed as the “inverse” of the volume reducing projection.

2.2.3 Tensor representation matter fields

Adding rank-two symmetric or antisymmetric tensor representation fields to a $U(N_c)$ Yang-Mills theory reduces the center symmetry from $U(1)^d$ to $(\mathbb{Z}_2)^d$ (corresponding to negating all link variables pointing in a given direction.) Starting from a single-site model, one may use this symmetry (plus gauge invariance) to define a volume enlarging projection which maps the theory to one on a periodic cubic lattice of size 2^d . More generally, if one starts with the theory on a periodic lattice of size L^d , one may define volume enlarging projections which double the length of one or more lattice directions — provided L is odd. But, for reasons discussed in the Appendix, the same approach does not work if L is even.¹⁴

However, just as with fundamental representation fields, if one considers a theory with N_f degenerate flavors of matter fields, then a $U(N_c)$ gauge theory with a global $U(N_f)$ flavor symmetry may be viewed as the zero (flavor) coupling limit of a $U(N_c) \times U(N_f)$ gauge theory — which does have a $U(1)^d$ center symmetry. And with this enlarged gauge group one may define projections which increase the volume by K^d , with K any integer, provided both N_c and N_f are divisible by K^d .¹⁵

2.3 Large N_c equivalence

In Refs. [6, 7], we proved an equivalence between the large N limits of a wide class of $U(N)$ field theories with adjoint matter and their orbifold projections yielding quiver gauge theories. More precisely, we demonstrated that the large- N dynamics of the parent and daughter

¹⁴In brief, if L is even then the result of the projection is not a theory on a larger lattice, but rather multiple decoupled copies of the theory on the original lattice. See the Appendix for details.

¹⁵However, with tensor representation matter fields it should be noted that one cannot define a natural large N_c limit by sending $N_c \rightarrow \infty$ with N_f/N_c held fixed. Consequently, this way of defining a volume enlarging projection will not be useful to us.

theories, within their respective neutral sectors, coincide.¹⁶ If the symmetries defining the neutral sectors are not spontaneously broken, then the ground states of parent and daughter theories will lie within their respective neutral sectors. In this case, the expectation values of corresponding single-trace neutral operators in parent and daughter theories will have coinciding large N limits. Moreover, the leading large- N behavior of *connected* correlators (as well as ground state or thermal free energies) are also directly related.¹⁷ It is important to bear in mind that this large- N equivalence applies only to observables within the respective neutral sectors of the parent and daughter theories.

Ref. [6] demonstrated this large- N equivalence by comparing generalized loop equations (or Schwinger-Dyson equations for correlators of gauge invariant operators) of parent and daughter theories. Ref. [7] used a more abstract, but more powerful, approach involving the comparison of large N classical dynamics constructed from suitable large N coherent states.¹⁸

The methods, and results, of Refs. [6, 7] extend immediately to the volume-reducing and volume-enlarging projections discussed above in gauge theories containing only adjoint representation matter fields. (The only difference is the precise form of the projection, and every step of the analysis of Refs. [6, 7] goes through with only minor cosmetic changes. Therefore, we refer readers to these references for details.) Consequently, the volume-reducing and volume-enlarging projections produce theories with coinciding large- N limits within their neutral sectors, just as discussed above.¹⁹ The corresponding situation for theories with fundamental or tensor representation fields is discussed separately below.

¹⁶In the parent theory, neutral states (or operators) are those which are invariant under the symmetries used to define the projection. In the daughter theory, neutral states (or operators) are those which are invariant under global symmetries in the daughter which are remnants of gauge symmetries in the parent theory. Non-neutral operators are also called “twisted”.

¹⁷If the projection maps the neutral single trace operators \mathcal{O}_i in the parent to corresponding operators $\tilde{\mathcal{O}}_i$ in the daughter, then

$$\lim_{|G_p| \rightarrow \infty} |G_p|^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}} = \lim_{|G_d| \rightarrow \infty} |G_d|^{M-1} \langle \tilde{\mathcal{O}}_1 \cdots \tilde{\mathcal{O}}_M \rangle_{\text{conn}}, \quad (2.15)$$

where $|G_p|$ and $|G_d|$ denote the dimension of the gauge group in the parent and daughter theory, respectively. Ground state energies (or free energies) satisfy the relation,

$$\lim_{|G_p| \rightarrow \infty} |G_p|^{-1} E_{\text{g.s.}}^{\text{parent}} = \lim_{|G_d| \rightarrow \infty} |G_d|^{-1} E_{\text{g.s.}}^{\text{daughter}}. \quad (2.16)$$

¹⁸The comparison of loop equations directly reveals the necessity of the symmetry realization conditions for large- N equivalence of the lattice regularized field theories, or their continuum limits. If the symmetries defining the neutral sectors are spontaneously broken, then the loop equations of the two theories do not coincide, invalidating equivalence. Conversely, unbroken symmetries defining the neutral sectors imply coinciding loop equations. However, because loop equations generally can (and do) have multiple solutions, a demonstration of coinciding loop equations does not constitute a proof of equivalence, except in the large coupling, large mass phase of the lattice regulated theory where one can prove that the loop equations have a unique physical solution. The coherent state approach of Ref. [7] plugs this loophole and demonstrates equivalence in any phase of the theories (including their continuum limits) which satisfy the necessary and sufficient symmetry realization conditions.

¹⁹For a $U(N_c)$ gauge theory on a lattice of size L^d , note that the actual gauge symmetry group is $U(N_c)^{L^d}$

If $\{\mathcal{O}_i\}$ are arbitrary gauge invariant single trace operators, with definite momenta which are commensurate with an L^d volume (*i.e.*, whose momentum components are integer multiples of $2\pi/L$), then the large- N equivalence applied to the projection from a volume $\Lambda = (KL)^d$ to a smaller volume $\Lambda' = (L)^d$ implies that connected correlators of these operators satisfy

$$\lim_{N \rightarrow \infty} (K^d N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{N, KL} = \lim_{N \rightarrow \infty} (N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{N, L}, \quad (2.17)$$

where $\langle \cdots \rangle^{N, L}$ denotes an expectation value in a periodic cube of size L with the specified value of N , *provided* the theory in volume Λ' does not spontaneously break the $(\mathbb{Z}_K)^d$ subgroup of the $U(1)^d$ center symmetry and the theory in volume Λ does not spontaneously break translation invariance (by multiples of L). Similarly, the large- N equivalence applied to the volume-enlarging projection from Λ' back to Λ implies that

$$\lim_{N \rightarrow \infty} (N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{N, KL} = \lim_{N \rightarrow \infty} (K^d N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{K^d N, L}, \quad (2.18)$$

again provided that $(\mathbb{Z}_K)^d$ center symmetry is not spontaneously broken in volume Λ' , and translation invariance is unbroken in volume Λ .

Volume-changing orbifold projections directly connect theories in volumes of commensurate size (*i.e.*, with spatial periods which are integer multiples of each other). Volume independence is more general, but this too follows from the large N orbifold equivalence, because the equivalence of large N dynamics (within respective neutral sectors) is a transitive relation among theories. For example, consider a $U(N_c)$ gauge theory formulated in volume $\Lambda = (KL)^d$ and in volume $\Lambda' = (K'L)^d$. Let Q be the greatest common divisor of K and K' , $Q = \text{gcd}(K, K')$. If K is not divisible by K' (or vice-versa), then no volume-changing projection directly relates these two volumes. However, a volume-enlarging projection relates both of these theories to volume $\Lambda'' = (K''L)^d$, with $K'' = KK'/Q$. Therefore, correlators of corresponding neutral observables in the theories in volume Λ and Λ' satisfy the appropriate generalization of Eq. (2.18), namely,

$$\lim_{N \rightarrow \infty} (K'^d N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{(K'/Q)^d N, KL} = \lim_{N \rightarrow \infty} (K^d N^2)^{M-1} \langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}^{(K/Q)^d N, K'L}, \quad (2.19)$$

provided the $(\mathbb{Z}_{K'/Q})^d$ center symmetry is not spontaneously broken in volume Λ , and the $(\mathbb{Z}_{K/Q})^d$ center symmetry is not broken in Λ' . This relation is applicable to single-trace

(since there is an independent $U(N_c)$ group on each site) with a total dimension of $L^d N_c^2$. For a volume-reducing projection mapping a parent $U(N_c)$ theory on a lattice Λ of size $(KL)^d$ to a daughter $U(N_c)$ theory on a lattice Λ' of size L^d , the factor in the relation (2.4) between the actions of these theories equals the ratio of the dimensions of their gauge groups, $|G_p|/|G_d| = K^d$. And for the volume-enlarging projection taking a $U(K^d N)$ theory on a size L^d lattice to a $U(N)$ theory on a size $(KL)^d$ lattice, the factor in the relation (2.11) between the actions in this case is also the ratio of the dimensions of their gauge groups, $|G_p|/|G_d| = [L^d (K^d N)^2]/[(KL)^d N^2] = K^d$. Therefore, these mappings between actions could have been written in either case as $|G_p|^{-1} S_{\text{parent}} \rightarrow |G_d|^{-1} S_{\text{daughter}}$, which is precisely the relation (2.16) between actions (or ground state energies) noted in footnote 17.

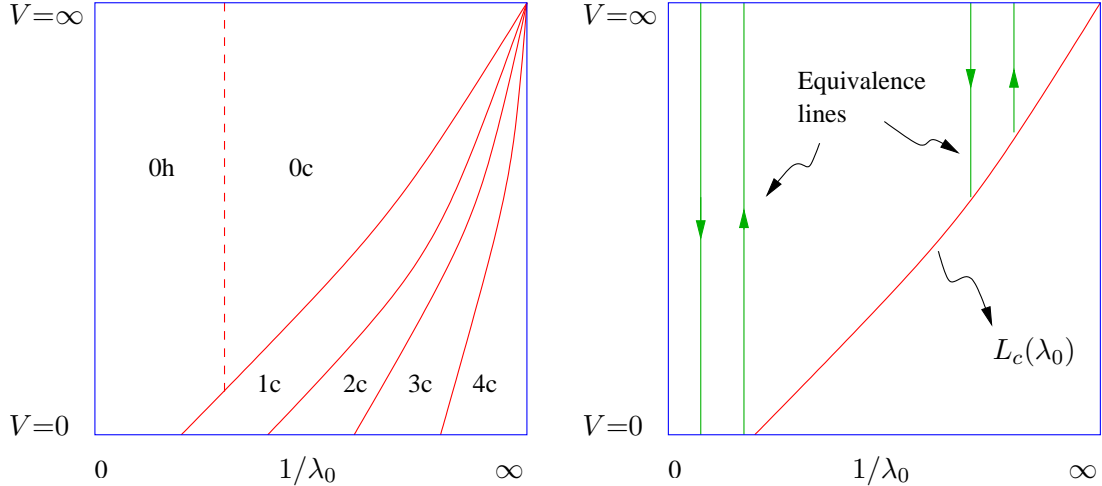


Figure 1: Left: the phase diagram of $U(N_c)$ lattice Yang-Mills theory on a four-dimensional periodic cube, in the limit of large N_c , as a function the lattice volume V and inverse (bare) 't Hooft coupling λ_0 . The $V = 0$ axis corresponds to the single-site Eguchi-Kawai model. Solid red lines separate phases with different realizations of the $U(1)^4$ center symmetry, as described in the text. These lines approach finite physical volumes (and infinite lattice volume) in the $\lambda_0 \rightarrow 0$ continuum limit. The dashed line is a lattice artifact and represents a Gross-Witten type phase transition in which the one plaquette eigenvalue distribution develops a gap. The figure is adapted from a description in Ref. [5]. Right: volume independence of large N lattice Yang-Mills theory, as a function of the inverse (bare) 't Hooft coupling. The vertical green lines are lines of large N equivalence. Both volume reduction (downward arrows) and volume expansion (upward arrows) may be interpreted as orbifold projections. The red line corresponds to the first center-symmetry breaking phase transition (separating the 1c phase from the 0c or 0h phases). Volume independence fails below this line. As discussed in the text, the addition of light adjoint representation matter fields can eliminate the existence of the phases with spontaneously broken center symmetry.

observables whose winding numbers vanish mod K'/Q in Λ (or mod K/Q in Λ'), and whose momenta are quantized in units of $2\pi/(QL)$.

Of course, the recognition of the volume independence of large N gauge theories is far from new and, in the case of pure gauge theories, goes back to the original paper of Eguchi and Kawai [1]. (The coherent state proof of equivalence, for pure gauge theories, is contained in a small comment in Ref. [3].) But the natural connection to orbifold projections, the appropriate extension to theories with matter, and the validity of the above relations for connected correlators²⁰ have not been widely appreciated.

Translation invariance is not expected to break spontaneously in any normal local field theory. But, as noted in the Introduction, it has long been known that for sufficiently weak coupling the single-site EK model (in more than two dimensions) spontaneously breaks the

²⁰The large- N equivalence of the Eguchi-Kawai matrix model (2.6) to lattice gauge theory only applies to neutral operators \mathcal{O}_i in the finite-volume theory with zero momentum. Therefore, information (such as glueball masses) encoded in long-distance behavior of correlations is not captured by the matrix model. Retaining such information in a reduced volume theory requires keeping at least one uncompactified dimension.

$U(1)^d$ center symmetry and therefore its correlators cease being equivalent (in the large N limit) to those of large volume Yang-Mills theory. For large- N pure gauge theory on larger periodic cubic lattices, the situation has been clarified in recent years thanks to the numerical work of Kiskis, Narayanan and Neuberger [4, 5]. The $d=4$ phase diagram they find, as a function the bare inverse 't Hooft coupling $1/\lambda_0$ and lattice volume V , is shown schematically in Fig. 1. The continuum limit corresponds to $1/\lambda_0 \rightarrow \infty$. The $U(1)^4$ center symmetry is broken spontaneously to $U(1)^3$, $U(1)^2$, $U(1)$, and nothing in the phases labeled 1c, 2c, 3c, and 4c, respectively. All symmetries are unbroken in the phases labeled 0h and 0c. These two phases are distinguished by the single plaquette eigenvalue distribution; in phase 0h it has support on the entire unit circle, while it develops a gap in phase 0c. The transition between these two phases is an analog of the Gross-Witten transition in two dimensions [28], and is not associated with spontaneous breaking of any symmetry. The symmetry breaking phase transition lines, appearing as red curved lines in the figure, approach scaling lines which reach finite physical volume (and infinite lattice volume) in the continuum limit [5].

Consequently, large- N volume independence of Yang-Mills theory (in $d > 2$) is valid only in the 0c and 0h phases, *i.e.*, above the first symmetry-breaking phase transition line, which defines the minimal size $L_c(\lambda_0)$ of a periodic lattice for which large N volume independence holds. This is illustrated on the right in Fig. 1. The addition of adjoint representation matter fields can affect the existence of a minimal size for which large N volume independence holds. In particular, one or more flavors of light adjoint representation fermions, defined with periodic boundary conditions, can entirely suppress the center-symmetry breaking phase transitions. This is discussed below in the section 3. In such theories, large- N volume independence holds for periodic cubic volumes of any size .

Once again, it must be emphasized that large- N orbifold equivalence applies only to suitable “neutral” observables. In the case of volume independence, this means single-trace observables whose momenta and winding numbers are commensurate with both volumes one is comparing. For the validity of large- N orbifold equivalence, what is significant is not the existence of non-neutral sectors, but rather the realization of the symmetries which define the relevant neutral sectors. This inevitably depends on the detailed dynamics of the theory, as well as the volume under consideration. The example of large- N volume independence, depicted in Fig. 1, is a particularly clear illustration of this point.

Fundamental representation matter

As noted earlier, for theories with fundamental representation matter fields and $U(N_f)$ flavor symmetry, one can define a volume-enlarging projection (for suitably large values of N_c and N_f) by regarding the flavor symmetry as weakly gauged and exploiting the $U(1)^d$ center symmetry which is part of the enlarged gauge group. If N_f/N_c is held fixed as $N_c \rightarrow \infty$, then the methods of Refs. [6, 7] also apply to the comparison of these theories in different volumes, and imply that large N dynamics coincides within their respective neutral sectors. There is,

however, one crucial difference affecting the utility of this result. In a $U(N_c) \times U(N_f)$ gauge theory, the center symmetry which defines the neutral sector of the smaller volume theory is *always* spontaneously broken in the limit of arbitrarily weak coupling for the flavor group. This merely reflects the fact that arbitrary constant, commuting values of the flavor gauge field are equally valid vacuum configurations in the limit of vanishing gauge coupling for the flavor group. Therefore large N volume independence does not hold for vacuum (or thermal) expectation values in theories with fundamental representation matter when N_f/N_c is held fixed.

This failure of large N_c volume independence may be understood more physically by considering the Casimir energy of these theories. In a confining theory, if one or more dimensions are compactified then the theory will have a non-zero Casimir energy whose value depends on the size of the compactification. In a pure gauge theory, the spectral density of glueballs remains $O(1)$ in the large N_c limit, and hence their contribution to the Casimir energy is also $O(1)$. Large N_c volume independence only applies to $O(N_c^2)$ contributions to the ground state energy, and this is unaffected by the volume dependent Casimir energy contributions. The same result holds for theories containing adjoint representation matter fields (whose number does not grow with N_c). But if the theory contains N_f degenerate fundamental representation matter fields, then the hadron spectrum will include bound mesons, and their spectral density will be proportional to N_f^2 (since one can independently specify the flavor of the valence quark and antiquark). Consequently, the Casimir energy will scale as N_f^2 , implying that large N_c volume independence (of ground state properties) must fail if N_f/N_c is held fixed.

If instead one chooses to hold N_f fixed as $N_c \rightarrow \infty$, then the fundamental representation fields have no effect whatsoever on the leading $O(N_c^2)$ contribution to the ground state energy (or free energy), or on the leading large N_c behavior of connected correlators of single-trace (gluonic) observables. However, one may consider subleading $O(N_c)$ contributions to the vacuum energy (or free energy), as well as the behavior of connected correlators involving mesonic operators. Large N_c volume independence *does* hold for $O(N_c)$ thermodynamic contributions, and for mesonic correlators, when N_f is held fixed. One may show this by examining the coherent state algebra for the mesonic sector [29,30] or, perhaps more directly, by integrating out the fundamental representation fermions and considering the resulting non-local gluonic observables. Under this procedure, the expectation value of a fermion bilinear $\bar{\psi}(x)M\psi(x)$ becomes $\text{tr}[MG(x,x)]$ where $G(x,y)$ is the Green's function for the Dirac operator \not{D} in an arbitrary background gauge field and M is some matrix acting on Dirac indices. This matrix element of the Dirac propagator may be expanded as a linear combination of Wilson loops passing through the point x . These will include both topologically trivial loops which lie in the neutral sector of a volume-enlarging projection, and topologically non-trivial loops with non-zero net winding around the periodic volume. However, as long as the center symmetry of the gluonic theory (in the absence of fundamental representation matter) is unbroken, then the expectation value of all topologically non-trivial loops, which lie in the

twisted sector, will vanish. So the expectation value of any gauge invariant fermion bilinear is equivalent to the expectation value of a neutral single-trace gluonic operator, provided the symmetry realization necessary for large N_c volume independence is satisfied.

The same argument can be applied to connected correlators of mesonic operators. Integrating out the fermions in this case will generate both expectation values of single trace contributions of the form

$$\text{tr}[M_1 G(x_1, x_2) M_2 G(x_2, x_3) M_3 \cdots G(x_M, x_1)], \quad (2.20)$$

and connected correlators of multi-trace contributions, such as

$$\text{tr}[M_1 G(x_1, x_1)] \text{tr}[M_2 G(x_2, x_3) M_3 \cdots G(x_M, x_2)]. \quad (2.21)$$

The single-trace contributions are linear combinations of Wilson loops. Once again, only the topologically trivial (neutral) loops, to which large N_c volume independence applies, have non-vanishing expectation values as long as the center symmetry of the gluonic theory is unbroken. But the multi-trace contributions lead to connected correlators of both topologically trivial and non-trivial Wilson loops. Connected correlators of topologically non-trivial Wilson loops are non-zero even when center symmetry is unbroken.

Despite the existence of these non-neutral (and volume dependent) contributions to connected correlators of mesonic operators, volume independence is valid for the leading large N_c behavior of connected correlators of arbitrary mesonic operators,²¹ when the center symmetry of the gluonic theory is unbroken and N_f is held fixed as $N_c \rightarrow \infty$. The key point is that the multi-trace contributions to connected correlators are suppressed by a power of N_f/N_c relative to the single trace contributions. If $\{\mathcal{O}_i\}$ are mesonic operators scaled to have finite large N_c expectation values, then standard large N_c counting [31] shows that the single-trace contributions to $\langle \mathcal{O}_1 \cdots \mathcal{O}_M \rangle_{\text{conn}}$ are $O(N_c^{-(M-1)})$. But because connected correlators of gluonic operators scale with powers $1/N_c^2$, not $1/N_c$, multi-trace contributions to connected correlators of mesonic operators are subdominant relative to the leading single-trace contributions, provided $N_f/N_c \ll 1$.

Tensor representation matter

An analysis of either loop equations, or large N coherent state dynamics, may be used to examine volume independence in large N_c theories with rank-two tensor representation matter fields. Appropriately adapting the methods of Refs. [6, 7], one finds that large N_c volume independence is valid, just as in theories with adjoint representation matter fields. In particular, Eqs. (2.17)–(2.19) expressing the equivalence between connected correlators (or expectation values) of neutral single trace observables in volumes $\Lambda = (KL)^d$ and $\Lambda' = (L)^d$ hold provided that translation invariance is not spontaneously broken in the larger volume theory, and the

²¹Or products of mesonic and neutral single-trace operators.

$(\mathbb{Z}_2)^d$ center symmetry is not spontaneously broken in the smaller volume theory.²² Just as with adjoint representation matter, this equivalence applies to single trace observables whose winding numbers, and total momentum, are compatible with the spatial periodicity in both volumes.

2.4 Temperature dependence

In our discussion of volume changing projections we have, purely for ease of presentation, focused on the case of mappings which uniformly affect all dimensions. However, all discussion of large N spacetime volume independence applies equally well to mappings which expand or contract different dimensions independently. The particular case of large N equivalence under mappings which change the size of just one periodic dimension is relevant for understanding temperature dependence in large N gauge theories (since a non-zero temperature theory is merely one in which a chosen Euclidean time direction has been compactified with a period equal to the inverse temperature $\beta \equiv 1/T$).

Consider some large N gauge theory at two different (inverse) temperatures β and β' . As always, large N equivalence applies only to observables in appropriate neutral sectors, and only if the symmetries defining these neutral sectors are not spontaneously broken. If $\beta = K\beta'$, for some integer K , then the neutral sector in the lower temperature theory consists of all single trace observables whose Matsubara frequencies are integer multiples of $2\pi/\beta'$ (not just integer multiples of $2\pi/\beta$). This, of course, is equivalent to the requirement that the observables be invariant under Euclidean time translation by multiples of β' . In the higher temperature theory, the neutral sector consists of all single trace observables whose winding numbers around the thermal circle are multiples of K . The large N limits of expectation values of corresponding neutral observables coincide, and their connected correlators satisfy

²²There are some interesting subtleties with tensor representation matter, related to the fact that the center symmetry is only $(\mathbb{Z}_2)^d$, not $U(1)^d$. In the loop equations for expectation values of winding number zero observables, the splitting terms arising from self-intersections can generate (after using large N factorization) the product of expectations of two smaller loops, each of which may have a non-zero winding number. Since the possible set of such non-zero winding number loops is volume dependent, if such (non-neutral) loops have non-zero expectation values, this violates large N_c volume independence. In $U(N_c)$ gauge theories with only adjoint representation matter, the $U(1)^d$ center symmetry, if unbroken, guarantees (for any value of N_c) that all loops with non-zero winding numbers have vanishing expectation values. But in theories with only a $(\mathbb{Z}_2)^d$ center symmetry [such as $U(N_c)$ theories with tensor representation matter fields, or $O(N_c)$ pure gauge theories], the center symmetry only forces loops with odd winding numbers to have vanishing expectation values. Loops with even winding numbers can, and will, have non-zero expectations for finite values of N_c . But these expectation values are $1/N_c$ suppressed in the large N_c limit, provided the $(\mathbb{Z}_2)^d$ center symmetry is unbroken. This may be understood from the loop equations for observables with non-zero but even winding numbers. It is the splitting terms which act like source terms that generate non-zero expectation values for larger loops. In, for example, the equation for a winding number two observable, the source terms can involve products of winding number one loops, but cannot have products of only winding number zero loops. As long as the $(\mathbb{Z}_2)^d$ center symmetry is unbroken, the disconnected part of the expectation of a product of two winding number one loops will vanish, while the connected part is $1/N_c$ suppressed. At $N_c = \infty$ there is, in essence, a topological conservation law which prevents non-zero winding number sectors from directly coupling to the zero winding number sector, even when the center symmetry, alone, would allow such coupling.

the relation (2.17) (with $d = 1$), *provided* the \mathbb{Z}_K symmetry defining the neutral sector in the higher temperature theory is not spontaneously broken.

In the context of thermal gauge theories with only adjoint or tensor representation matter, the realization of the center symmetry associated with the thermal circle provides a sharp distinction between a confining phase, in which the center symmetry is unbroken, and a deconfined plasma phase, in which the center symmetry is spontaneously broken. Hence, the condition of unbroken center symmetry, required for large N volume independence, is precisely the same as the requirement that the theory be in a confining phase.

Therefore, within a confining phase, large N_c volume independence implies that the $O(N_c^2)$ part of the free energy is temperature independent. This is true, but trivial, in pure gauge theories, and theories with only adjoint or tensor representation matter fields, since temperature dependent contributions to the free energy in the confining low temperature phase of such theories only arises from glueball-like bound states whose spectral density is $O(N_c^0)$.²³

The implications of large N temperature independence are non-trivial for expectation values and connected correlators of neutral operators. If β and β' are not rationally related, then the neutral sector consists of single-trace observables for which both the Matsubara frequency and winding numbers around the thermal circle vanish.²⁴ Large N temperature independence implies, for example, that the value of the chiral condensate in QCD (with N_f fixed), or in $\mathcal{N} = 1$ super-Yang-Mills, is temperature independent at $N_c = \infty$ in the confining phase. The non-perturbative gluon condensate $\langle F_{\mu\nu}^2 \rangle$ and the area law coefficient of spacelike Wilson loops must likewise be temperature independent. But large N temperature independence does not apply to the physical string tension, since this is determined by the correlator of Wilson lines which wrap just once around the thermal circle, and these operators do not lie within the relevant neutral sector.

3. Eguchi-Kawai reduction for QCD

As reviewed above (and illustrated in Fig. 1), large N_c volume independence for pure Yang-Mills theory (in $d > 2$ dimensions) is valid only above a critical size, $L > L_c(d)$. Below this limit, spontaneous breaking of the $U(1)^d$ center symmetry invalidates volume independence. Adding fundamental representation matter fields either has no effect on this breakdown of

²³Temperature independence of the free energy also applies to the $O(N_c)$ part of the free energy if there are fundamental representation matter fields with N_f held fixed as $N_c \rightarrow \infty$, but this temperature independence is again trivial since the thermal contribution of mesons is $O(N_f^2)$ which, by assumption is $O(N_c^0)$. However, if N_f scales with N_c , so that their ratio is held fixed, then the free energy does have non-vanishing $O(N_c^2)$ temperature dependence and large N temperature independence fails to hold.

²⁴If β and β' are rationally related, so that $\beta/M = \beta'/M'$ for coprime integers M and M' , then the neutral sector is larger and consists of observables whose Matsubara frequencies are multiples of $2\pi/(\beta/M)$, and whose winding number in the temperature β theory vanish modulo M' (or equivalently whose winding number in the β' theory vanish modulo M).

large N_c volume independence (if N_f is held fixed), or completely destroys volume independence (if N_f/N_c is held fixed) for large volumes.

The effect of adjoint or tensor representation matter fields on volume independence is more interesting, since a single flavor of such matter fields can have a significant impact on the dynamics of the theory, even at $N_c = \infty$. If one considers QCD(AS) [or QCD(S)] with one or more compactified dimensions, then a simple perturbative analysis of the one-loop effective potential for Wilson lines [13, 17], which is reliable when the compactification radius is small compared to $\Lambda_{\text{QCD}}^{-1}$, shows that the $(\mathbb{Z}_2)^d$ center symmetry is spontaneously broken.²⁵ (This is true regardless of whether one uses periodic or antiperiodic boundary conditions for the fermions.) But a confining phase with unbroken center symmetry (and spontaneously broken chiral symmetry) will surely exist in sufficiently large volume. Therefore, large N_c volume independence in QCD(AS/S) fails below some critical size, which must be of order $\Lambda_{\text{QCD}}^{-1}$, just as in pure Yang-Mills theory.

The situation with adjoint representation fields is different. Consider $U(N_c)$ QCD(Adj) with one or more flavors of massless adjoint Majorana fermions. (Asymptotic freedom requires that $N_f \leq 5$.) Let one dimension be compactified with a radius L which is much smaller than all other dimensions, as well as $\Lambda_{\text{QCD}}^{-1}$, and choose periodic boundary conditions for the adjoint fermions. So the theory is on $\mathbf{R}^3 \times S^1$, where the S^1 may be regarded as a spatial circle. The one-loop effective potential for the Wilson line around this compactified direction is²⁶

$$\begin{aligned} V_{\text{eff}}^{\text{QCD(Adj)}}(\Omega) &= (N_f - 1) \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} |\text{tr } \Omega^n|^2 \\ &= \frac{N_f - 1}{24\pi^2 L^4} \left\{ \frac{8\pi^4}{15} N_c^2 - \sum_{i,j=1}^{N_c} [v_i - v_j]^2 ([v_i - v_j] - 2\pi)^2 \right\}. \end{aligned} \quad (3.1)$$

Here $\{e^{iv_j}\}$ denote the eigenvalues of the (untraced) Wilson line Ω , and $[x] \equiv x \bmod 2\pi$ indicates quantities defined to lie within the interval $[0, 2\pi)$. Note that this effective potential is positive for $N_f > 1$, negative for $N_f = 0$, and vanishes for $N_f = 1$. If $N_f = 0$, then the potential generates an attractive interaction between eigenvalues leading to spontaneous breaking of the center symmetry and a non-zero expectation value for the Wilson line. This is just the pure Yang-Mills case discussed above, in which center symmetry breaking invalidates volume independence in small volumes.

If $N_f > 1$, then the potential generates a repulsive interaction between eigenvalues. Consequently, the Wilson line eigenvalues distribute uniformly around the unit circle, the Wilson line expectation value $\langle \text{tr } \Omega \rangle$ vanishes, and the center symmetry is unbroken.²⁷

²⁵The small volume phase also has unbroken chiral symmetry. See Ref. [17] for further details. This analysis requires that $1 \leq N_f \leq 5$. Adding more than five flavors destroys asymptotic freedom.

²⁶A completely analogous, but more complicated, formula results if one compactifies two or more dimensions with radius L . The following discussion is unaffected. See Ref. [13] for details.

²⁷As noted above, we are viewing the S^1 as a spatial circle, and accordingly the relevant center symmetry

If $N_f = 1$, then the massless QCD(Adj) theory is just $\mathcal{N}=1$ supersymmetric Yang-Mills theory. For this case, the one-loop potential identically vanishes. Supersymmetry guarantees that this remains true at all loop orders. However, there is an instanton induced non-perturbative superpotential [32]. The role of the non-perturbative potential in the $N_f = 1$ case is essentially identical to the role of the one-loop effective potential when N_f is greater than one. It also provides eigenvalue repulsion which leads to unbroken center symmetry and vanishing Wilson line expectations, $\langle \text{tr } \Omega \rangle = 0$. The instanton analysis is justified in sufficiently small volume, but the simple dependence on the holomorphic coupling required by supersymmetry guarantees that the result remains valid in arbitrary volume.

Consequently, QCD with $1 \leq N_f \leq 5$ massless adjoint representation fermions (and periodic boundary conditions) does not undergo a spatial center symmetry breaking phase transition in small volume.²⁸ If non-zero fermion masses are introduced, this symmetry realization will be stable provided the fermion masses are small compared to Λ_{QCD} .²⁹ Therefore, with at least one (and at most five) light fermions, QCD(Adj) is an asymptotically free, confining gauge theory which satisfies large N_c volume independence all the way down to zero size.

To recap, large N_c volume independence fails for sufficiently small volumes in QCD(AS/S) but remains valid in QCD(Adj) (with periodic boundary conditions for fermions). Despite this difference, the validity of volume independence down to zero size in QCD(Adj) has interesting implications for QCD(AS/S). The key ingredient for this connection is the existence of a large N_c equivalence [10, 11, 17] between QCD(AS/S) and QCD(Adj) (for coinciding values of N_f , 't Hooft coupling, fermion masses, and boundary conditions) provided that charge conjugation symmetry (\mathcal{C}) is not spontaneously broken in QCD(AS/S).³⁰ The appropriate neutral sector in QCD(AS/S) for this large N_c equivalence consists of charge conjugation even single-trace bosonic operators. Consequently, as long as charge conjugation symmetry is not spontaneously broken in QCD(AS/S), expectation values and connected correlators of corresponding \mathcal{C} -even single-trace operators in QCD(AS/S) and QCD(Adj) will coincide at

is a spatial center symmetry. Its realization should not be interpreted as a confinement criterion, since it is temporal Wilson lines, not spatial Wilson lines, which are order parameters for confinement/deconfinement transitions. But if antiperiodic boundary conditions are chosen, then one may instead regard the S^1 as a temporal circle, in which case a small radius compactification is equivalent to the high temperature limit. For this choice, regardless of the number of fermions, the one-loop potential generates an attractive interaction between the Wilson line eigenvalues which causes spontaneous breaking of the (temporal) center symmetry, thereby signaling deconfinement and invalidating large N_c temperature independence.

²⁸This is true for $SO(2N_c)$ and $Sp(2N_c)$ gauge groups as well as for $U(N_c)$.

²⁹But as one raises the fermion mass, in sufficiently small volumes, there will be a phase transition to a center-symmetry broken phase at some critical mass of order Λ_{QCD} , since the behavior of the theory must approach that of pure Yang-Mills theory when all fermion masses are large compared to the strong scale.

³⁰This so-called “orientifold equivalence” is an example of a daughter-daughter orbifold equivalence; one can obtain either $U(N_c)$ QCD(AS/S) or $U(N_c)$ QCD(Adj) by applying different orbifold projections to a common parent theory [namely, QCD(Adj) with either $SO(2N_c)$ or $Sp(2N_c)$ gauge group]. See Ref. [17] for further discussion.

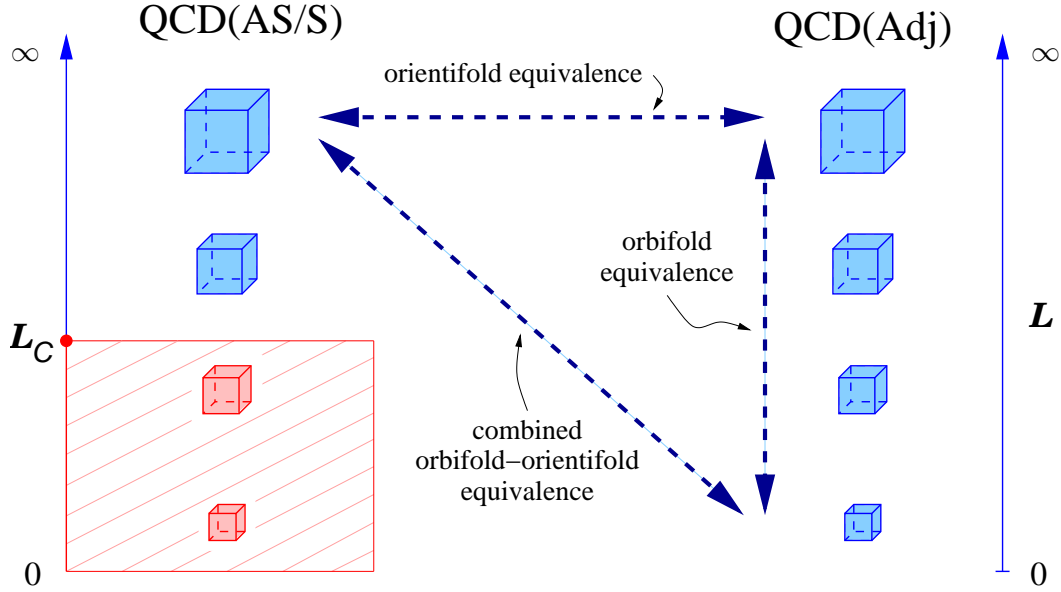


Figure 2: Large N_c equivalences relating QCD(AS/S) and QCD(Adj), as a function of the size L of the periodic volume in which the theories are defined. Provided periodic boundary conditions are used for fermions, volume independence holds in QCD(Adj) for all L . But in QCD(AS/S), volume independence fails below a critical size, $L < L_C$, (shaded region) due to spontaneous breaking of center symmetry. This prevents reduction all the way down to a single-site matrix model for QCD(AS/S). Large N_c orientifold equivalence holds between QCD(Adj) and QCD(AS/S) as long as charge conjugation symmetry is unbroken in QCD(AS/S). This should be true in sufficiently large volumes, but fails when $L < L_C$. The combination of volume changing orbifold projections in QCD(Adj) along with the orientifold equivalence in large volume provides a useful equivalence between small volume QCD(Adj) and large volume QCD(AS/S). In particular, a single-site matrix model of QCD(Adj) will reproduce properties of infinite volume QCD(AS/S).

$$N_c = \infty.$$

In sufficiently small volume, charge conjugation symmetry *is* spontaneously broken in QCD(AS/S) if one uses periodic boundary conditions for fermions. In this case, the spontaneous breaking of the $(\mathbb{Z}_2)^d$ center symmetry mentioned above is associated with the development of an imaginary expectation for the Wilson line, which indicates spontaneous breaking of charge conjugation (as well as parity and time reversal) symmetry, in addition to the broken spatial center symmetry [17]. This symmetry breaking is sensitive to the choice of boundary conditions for the fermions and is evidently a finite size effect. There is no known reason to expect spontaneous breaking of charge conjugation symmetry in sufficiently large volumes. For the case of $N_c = 3$ and $N_f = 4$, recent lattice simulations [18] have clearly seen the presence of a small volume phase in QCD(AS) with spontaneously broken charge conjugation symmetry, together with clear evidence for a phase transition consistent with restoration of

charge conjugation symmetry at a critical size very near the inverse strong scale. [The chiral symmetry realization also changes at this transition, indicating the presence of a chirally symmetric phase of QCD, at zero (or low) temperature, in sufficiently small volume.] Therefore, it seems reasonable to believe that QCD(AS/S) does have unbroken charge conjugation symmetry in large volume.³¹

Consequently, the large N_c equivalence between QCD(AS/S) and QCD(Adj) should be valid above a critical size, while large N_c volume independence in QCD(Adj) is valid for all sizes. Hence, even though volume independence fails in QCD(AS/S) in sufficiently small volume, *QCD(AS/S) in large volume has a large N_c equivalence relating it to QCD(Adj) in arbitrarily small volumes.* This is depicted schematically in Fig. 2.

This equivalence applies to observables in the intersection of the neutral sectors for the orientifold equivalence and volume changing orbifold projections. Explicitly, these are \mathcal{C} -even single-trace observables whose total momenta are compatible with the small volume. In particular, a single-site version of QCD(Adj), which is just the EK matrix model (2.6) augmented with adjoint representation Grassmann variables, will reproduce the leading large N_c behavior of all expectation values, and zero-momentum connected correlators, of arbitrary \mathcal{C} -even single-trace observables in infinite volume QCD(AS) — which is a natural generalization of real QCD to large N_c . This is a version of Eguchi-Kawai reduction of large N_c QCD which works all the way down to zero size.

4. Theory space volume independence

The spacetime volume independence discussed in the previous section has a natural counterpart involving large N equivalences among quiver gauge theories. It is convenient to represent the gauge group and matter content of such theories by a “theory space graph” (or “quiver diagram”), as illustrated in Fig. 3. Each node represents a simple gauge group factor, while each bond represents a matter field. A directed bond which connects two nodes represents a matter field transforming under the fundamental representation of the gauge group factor where it starts, and the anti-fundamental representation of the gauge group factor where it ends. A bond which begins and ends at the same node represents a matter field transforming under the adjoint representation of the corresponding gauge group factor. Note that in any quiver theory containing only adjoint or bifundamental matter fields, gauge invariant single-trace operators must correspond to closed loops in theory space. (For more detailed discussion see, for example, Refs. [22, 23].)

³¹However, there is no rigorous proof of this. The Vafa-Witten theorem on \mathbf{R}^4 [25] demonstrates unbroken parity, but does not determine the charge conjugation symmetry realization. In fact, the theorem only rules out cases of spontaneous breaking of parity with a local order parameter. In a compactified theory, breaking of parity and center symmetry can be entangled so that order parameters must involve topologically non-trivial Wilson lines. Spontaneous symmetry breaking in this case is not ruled out by the Vafa-Witten theorem [and occurs in just this manner in QCD(AS/S)].

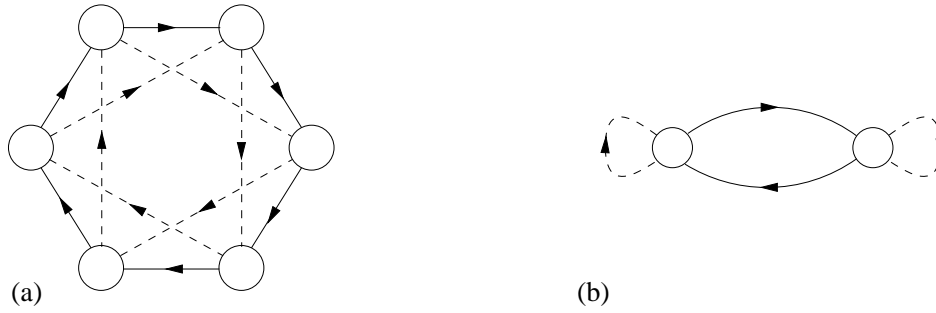


Figure 3: Examples of theory space graphs. Graph (a) shows the case of a $U(N)^6$ gauge theory with 12 bifundamental representation matter fields; one set, represented by solid bonds, transforms under “nearest neighbor” group factors while the other set, represented by dashed bonds, transforms under next-nearest neighbor group factors. This graph (and the underlying theory) has an obvious \mathbb{Z}_6 symmetry corresponding to cyclic permutations of the different group factors. Graph (b) shows the case of a $U(N)^2$ theory which may be obtained from theory (a) by applying a projection based on the \mathbb{Z}_3 subgroup of the theory space symmetry group.

4.1 Volume reducing projections

If the theory has a global symmetry which interchanges or permutes equivalent gauge group factors, then this symmetry will be a geometric invariance of the theory space graph. A simple example is a parent theory with a $U(N)^{L^d}$ gauge group and $N_f L^d$ bifundamental matter fields coupling the different gauge group factors in such a way that the resulting theory has a $(\mathbb{Z}_L)^d$ symmetry which cyclically permutes gauge group factors. Fig. 3(a) illustrates a case of $L = 6$, $d = 1$, and $N_f = 2$.

Let \mathcal{T} be some chosen subgroup of the global symmetries involving theory space permutations. One may define a projection which eliminates degrees of freedom that are not invariant under the subgroup \mathcal{T} . This is equivalent to identifying nodes (and bonds) of the theory space graph which are related by transformations in \mathcal{T} . The result will be a daughter theory with a theory space which is smaller than that of the parent, as illustrated in Fig. 3.

In the example of a parent theory with $(\mathbb{Z}_L)^d$ theory space symmetry, consider the case of $\mathcal{T} = (\mathbb{Z}_K)^d$, where L is divisible by K . The projection will yield a daughter theory with a smaller $(\mathbb{Z}_{L'})^d$ theory space, with $L' \equiv L/K$. The discussion in section 2.1, concerning projections based on a subgroup of the spacetime translation symmetry group, may be reapplied almost verbatim to this class of projections which reduce the size of theory space.

The neutral sector of the parent theory consists of gauge invariant single-trace operators which are invariant under the projection subgroup \mathcal{T} . The neutral sector of the daughter theory consists of gauge invariant single-trace operators whose winding numbers around the periodic cycles in theory space all vanish modulo K . The projection defines a one-to-one mapping between these two neutral sectors.

Once again, the action (or free energy) of the parent and daughter theories are related

by an overall factor of the size of the projection subgroup,

$$S_{\text{parent}} \longrightarrow K^d S_{\text{daughter}} . \quad (4.1)$$

4.2 Volume enlarging projections

One may also define projections which increase the size of theory space, in a manner completely analogous to the previous discussion of spacetime volume enlarging projections. The only difference between the two cases is that for quiver theories, it is the bifundamental matter fields which transform non-trivially under two different gauge group factors, while for a lattice gauge theory it is the link variables which transform non-trivially under gauge group factors located at two different sites.

The most commonly discussed examples of orbifold projections map a $U(N_c)$ theory with a simple gauge group into a quiver theory. (See, for example, Refs. [6, 22, 23].) These cases correspond precisely to the volume enlarging projections of the single-site Eguchi-Kawai model discussed above. A $(\mathbb{Z}_K)^d$ projection, for example, corresponds to constraints of the form

$$A_\mu = \gamma_\nu A_\mu \gamma_\nu^\dagger, \quad \nu = 1, \dots, d, \quad (4.2)$$

for the gauge fields (viewed as $N_c \times N_c$ matrices), together with

$$\Phi_a = \gamma_\nu \Phi_a \gamma_\nu^\dagger e^{2\pi i r_\nu [\Phi_a]/K}, \quad \nu = 1, \dots, d, \quad (4.3)$$

for matter fields. Here N_c must be divisible by K^d and $\{\gamma_\nu\}$ are the same mutually commuting matrices which appeared in Eq. (2.9) and are defined explicitly in Eq. (A.2).

Each matter field Φ_a is assigned a d -dimensional integer-valued charge vector with components $r_\nu[\Phi_a]$. Different choices for the charge vector assigned to each field correspond to different embeddings of $(\mathbb{Z}_K)^d$ into the product of the gauge and flavor symmetry groups, and produce daughter theories with differing connectivities in their theory space graphs. If the charge vector assigned to a particular field vanishes, then field becomes an adjoint representation field in the daughter theory. If the charge is non-zero, then the field becomes a bifundamental, with the value of the charge vector determining the connectivity of the corresponding bonds in the theory space graph.

The neutral sector in the parent $U(N_c)$ theory consists of gauge invariant single-trace operators for which the sum of the charge vectors of all matter field insertions vanish. The neutral sector in the daughter $[U(N)]^{K^d}$ theory consists of gauge invariant single-trace operators which are invariant under $(\mathbb{Z}_K)^d$ cyclic permutations of the gauge group factors (corresponding to translations in the periodic theory space graph). Once again, the projection defines a one-to-one mapping between these two neutral sectors, with the appropriate relation between the actions of the two theories involving a rescaling by the size of the projection,

$$S_{\text{parent}} \longrightarrow K^d S_{\text{daughter}} . \quad (4.4)$$

Instead of starting from a parent theory with a simple gauge group, one may define analogous theory-space enlarging projections starting from any quiver theory. This is done analogously to the volume expansion procedure of the lattice gauge theory, as discussed in the Appendix.

4.3 Large N equivalence

The entire discussion of large N equivalence in section 2.3 applies equally well to the above examples of theories related by $(\mathbb{Z}_K)^d$ theory-space enlarging or reducing projections. The $N = \infty$ dynamics within corresponding neutral sectors coincides. If the ground (or equilibrium) states of both theories lie within their respective neutral sectors, then the large N equivalence implies coinciding expectation values of corresponding neutral single-trace operators, and connected correlators of such operators related by Eqs. (2.17) and (2.18) (with $\langle \dots \rangle^{N,L}$ now denoting an expectation in a $[U(N)]^{L^d}$ quiver theory with a $(\mathbb{Z}_L)^d$ invariant theory space).

The condition that the ground (or equilibrium) state of a theory lie within its neutral sector is precisely the requirement that the symmetries defining the neutral sector not be spontaneously broken. The symmetry realization will, inevitably, depend on the specific dynamics of a theory. This includes not just the field content of the theory plus masses and couplings of matter fields, but also, for example, the temperature and the spatial volume in which the theory is defined (as well as the associated boundary conditions).

The most well-studied example of large N equivalence, the \mathbb{Z}_2 projection of $\mathcal{N} = 1$ supersymmetric $U(2N)$ Yang-Mills theory, yielding a non-supersymmetric $U(N) \times U(N)$ theory with a bifundamental fermion, turns out to be remarkably similar to the case of spacetime volume independence in pure Yang-Mills theories discussed above. For this \mathbb{Z}_2 projection of super-Yang-Mills, a useful large N equivalence requires that the \mathbb{Z}_2 symmetry exchanging gauge group factors in the non-supersymmetric quiver theory be unbroken. However, this symmetry is known to be spontaneously broken when the theory is compactified (with periodic boundary conditions) on $\mathbf{R}^3 \times S^1$, with the radius of the S^1 small compared to the inverse confinement scale, $R \ll \Lambda^{-1}$ [9]. In large volume, there is no evidence that this \mathbb{Z}_2 symmetry is broken, nor is there any proof (or solid evidence) that it is unbroken [12]. At sufficiently high temperature, $T \gg \Lambda$, it is clear that this \mathbb{Z}_2 symmetry is unbroken [17]. Hence, large N equivalence to $\mathcal{N} = 1$ super-Yang-Mills is valid at sufficiently high temperatures, and may be valid at low temperatures in sufficiently large volume.

5. Discussion

This paper has attempted to highlight the direct connection between volume (and temperature) independence in large N gauge theories and large N equivalences involving orbifold projections and quiver gauge theories. It is curious that nearly all discussion in the literature

concerning large N orbifold equivalence has focused on projections which increase the size of theory space, while nearly all previous discussion regarding large N spacetime volume dependence has focused on volume reducing mappings. As we have emphasized, one can easily define projections which either increase or decrease spacetime volume, or “theory space” volume. This unified view makes clear the common origin of the symmetry realization conditions which are necessary (and sufficient) for useful large N equivalence in all these examples.

Combining the large N_c volume independence of QCD with light adjoint fermions, valid down to arbitrarily small size (when periodic boundary conditions are used for fermions), with the large N_c “orientifold equivalence” relating QCD with adjoint and antisymmetric tensor representation fermion, which is valid in sufficiently large volumes, produces a large N_c equivalence between a single site model with adjoint fermions, and a natural large N_c generalization of real QCD. Although this equivalence only applies to charge conjugation even observables, it should nevertheless have practical utility for investigations of large N_c QCD. In particular, this form of a large N_c reduced model is applicable to both expectation values and suitable connected correlation functions (in contrast to quenched and twisted reduced models). Although simulations with dynamical fermions are always more challenging than pure gauge simulations, simulating a single site model with adjoint representation fermions should be relatively straightforward.

Many generalizations of the specific examples discussed above are possible. One natural generalization involves consideration of $O(N_c)$ or $USp(N_c)$ gauge theories instead of $U(N_c)$. We discuss the interconnections between such theories in a companion paper [24]. In quiver gauge theories, one may also consider theory-space enlarging projections based on non-Abelian, or non-freely acting, symmetry groups. As noted in the Introduction, for general projections based on arbitrary discrete symmetry groups, it is not clearly established when such projections lead to large N equivalence, and when they do not. But every case of large N equivalence that we are aware of is consistent with the conjecture that pairs of theories connected by “invertible” projections imply a large N_c equivalence between the two theories. (This conjecture is stated more formally in the Introduction.) Future work will hopefully shed more light on this issue of a global understanding of equivalences between different large N gauge theories.

Acknowledgments

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A. More on volume expansion as an orbifold projection

The main text discussed volume-expanding orbifold projections starting from single-site models. In this appendix we discuss the more general case of projections which will map a parent theory defined on a periodic lattice of any size to a daughter theory defined on a lattice which is larger by some (integer) factor $K > 1$. The lattice in question can be either the real space lattice on which the theory is defined, or the theory space of a quiver gauge theory. For simplicity of presentation, we will focus on pure gauge theories defined on a real space lattice, but what follows equally applies to volume expansion in theory space of quiver gauge theories.

Start with a parent lattice gauge theory on lattice $\Lambda' = (\mathbb{Z}_{L'})^d$ with gauge group $U(N_c)$, where $N_c = M^d N$ for some integer $M > 1$. The global symmetry group of the theory is $U(M^d N) \times U(1)^d$, where the first factor is the global part of the gauge symmetry group and the $U(1)^d$ is the center symmetry which phase rotates all link variables in a given direction, $U_\mu[\mathbf{n}'] \rightarrow \exp(i\theta_\mu) U_\mu[\mathbf{n}']$. Choose a cyclic subgroup $(\mathbb{Z}_M)^d \subset U(M^d N) \times U(1)^d$ of the global symmetry group and set all $(\mathbb{Z}_M)^d$ -noninvariant fields of the theory to zero. The appropriate embedding is the one for which the net effect of the projection is the imposition of the constraints

$$U_\mu[\mathbf{n}'] = \omega^{\delta_{\mu\nu}} \gamma_\nu U_\mu[\mathbf{n}'] \gamma_\nu^{-1}, \quad \mu, \nu = 1, \dots, d. \quad (\text{A.1})$$

Here, the matrices $\gamma_\nu \in U(M^d N)$ generate a $M^d N$ -dimensional representation of $(\mathbb{Z}_M)^d$ and are chosen as

$$\gamma_\nu = \underbrace{1_M \times \dots \times 1_M}_{\nu-1} \times \Omega \times \underbrace{1_M \times \dots \times 1_M}_{d-\nu} \quad (\text{A.2})$$

where 1_N and 1_M are $N \times N$ and $M \times M$ unit matrices, respectively, and

$$\Omega \equiv \text{diag}(1, \omega, \omega^2, \dots, \omega^{M-1}) \quad (\text{A.3})$$

with $\omega = e^{2\pi i/M}$. The phase factor $\omega^{\delta_{\mu\nu}}$ in Eq. (A.1) reflects a transformation under the $U(1)^d$ part of the symmetry group. This particular choice of the charge for the link fields guarantees that the daughter theory will retain nearest neighbor interactions on an enlarged lattice. [Adjoint representation matter fields in the theory would be neutral under the $U(1)^d$ symmetry, and hence would have no additional phase factor in their orbifold constraints.]

In order to find the form of link matrices $U_\mu[\mathbf{n}']$ which satisfy the orbifold constraint (A.1), it is helpful to introduce $M^d N \times M^d N$ “translation matrices” T_μ defined as

$$T_\mu = \underbrace{1_M \times \dots \times 1_M}_{\mu-1} \times S \times \underbrace{1_M \times \dots \times 1_M}_{d-\mu}, \quad (\text{A.4})$$

where S is a $M \times M$ “shift matrix” whose entries (defined modulo M) are $S_{ab} = \delta_{a, b-1}$. Different translation matrices commute with each other, $[T_\mu, T_\nu] = 0$. Using the definition of

the matrices γ_ν , one finds that the T_μ satisfy the same orbifold constraint (A.1), namely

$$T_\mu = \omega^{\delta_{\mu\nu}} \gamma_\nu T_\mu \gamma_\nu^{-1} . \quad (\text{A.5})$$

Thus if one defines $U_\mu[\mathbf{n}'] = \tilde{U}_\mu[\mathbf{n}'] T_\mu$, then the redefined matrices $\tilde{U}_\mu[\mathbf{n}']$ satisfy a simpler “neutral” constraint

$$\tilde{U}_\mu[\mathbf{n}'] = \gamma_\nu \tilde{U}_\mu[\mathbf{n}'] \gamma_\nu^{-1} , \quad (\text{A.6})$$

without any additional phase factor. In other words, all redefined link matrices $\tilde{U}_\mu[\mathbf{n}']$ must commute with the projection matrices $\{\gamma_\nu\}$. To find the most general solution of these constraints, it is helpful to note that every $M \times M$ matrix which commutes with Ω must be diagonal, and hence may be expressed as an order M polynomial in Ω . Consequently, any link variable $\tilde{U}_\mu[\mathbf{n}']$ that commutes with all γ_ν 's must have the form

$$\tilde{U}_\mu[\mathbf{n}'] = \sum_{\mathbf{p}} \Omega^{p_1} \times \dots \times \Omega^{p_d} \times \tilde{V}_\mu[\mathbf{n}', \mathbf{p}] \quad (\text{A.7})$$

where $\mathbf{p} = (p_1, \dots, p_d)$ is a vector whose components are integers running from 0 and $M-1$ (defined modulo M). This is a mixed representation in which \mathbf{n} is a real space label but \mathbf{p} is a momentum space vector lying in a Brillouin zone. It is far more convenient to express $\tilde{U}_\mu[\mathbf{n}']$ in a “position basis” via the discrete Fourier transform,

$$\tilde{V}_\mu[\mathbf{n}', \mathbf{p}] = \frac{1}{M^d} \sum_{\mathbf{m}} V_\mu[\mathbf{n}', \mathbf{m}] \bar{\omega}^{\mathbf{p} \cdot \mathbf{m}} \quad (\text{A.8})$$

where $\bar{\omega} = e^{-2\pi i/M}$ is complex conjugate of ω , and \mathbf{m} is a d -dimensional vector whose components are integers ranging from 0 to $M-1$ (modulo M). Then a matrix $\tilde{U}_\mu[\mathbf{n}']$ that solves the neutral constraint (A.6) has the form

$$\tilde{U}_\mu[\mathbf{n}'] = \sum_{\mathbf{m}} \Delta_{\mathbf{m}} \times V_\mu[\mathbf{n}', \mathbf{m}] , \quad (\text{A.9})$$

where the “basis matrices” $\Delta_{\mathbf{m}}$ are given by

$$\Delta_{\mathbf{m}} \equiv \frac{1}{M^d} \sum_{\mathbf{p}} \Omega^{p_1} \times \dots \times \Omega^{p_d} \bar{\omega}^{\mathbf{p} \cdot \mathbf{m}} , \quad (\text{A.10})$$

and each $V_\mu[\mathbf{n}', \mathbf{m}]$ is an arbitrary $N \times N$ unitary matrix. The basis matrices are mutually orthogonal projectors and satisfy

$$\text{tr } \Delta_{\mathbf{m}} = 1 , \quad \Delta_{\mathbf{m}} \Delta_{\mathbf{m}'} = \delta_{\mathbf{m}\mathbf{m}'} \Delta_{\mathbf{m}} , \quad \sum_{\mathbf{m}} \Delta_{\mathbf{m}} = 1_{M^d} . \quad (\text{A.11})$$

If the matrix $\tilde{U}_\mu[\mathbf{n}']$ is viewed as a collection of $M^d \times M^d$ blocks (each of which is $N \times N$) then only M^d of those blocks (labeled by \mathbf{m}) survive the constraint.

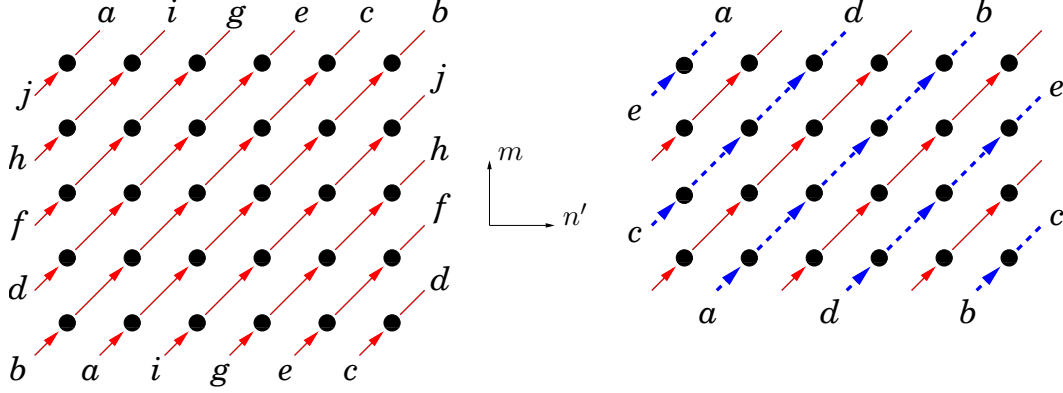


Figure 4: Examples of one-dimensional daughter theory lattices generated by \mathbb{Z}_5 (left figure) or \mathbb{Z}_4 (right figure) projections acting on a parent theory defined on a six site lattice (with \mathbb{Z}_6 translation symmetry). Each site in the daughter lattice has been labeled by two integers n' and m (even though these are one-dimensional lattices). Nearest-neighbor relations are defined by incrementing both n' and m by one. The \mathbb{Z}_5 projection yields a 30 site lattice with \mathbb{Z}_{30} translation symmetry. The path formed by the nearest-neighbor steps depicted as red arrows forms a single cycle around the daughter lattice. (Labels at the edge of the lattice show periodic identifications.) In contrast, a \mathbb{Z}_4 projection acting on the same parent theory yields two decoupled copies of a twelve site lattice with \mathbb{Z}_{12} translation symmetry. In this case, no sequence of nearest-neighbor steps can connect the two copies of the resulting daughter lattice.

With matrices $\tilde{U}_\mu[\mathbf{n}']$ of the form (A.9), it is not difficult to show that the Wilson action of the $U(N_c)$ lattice gauge theory on Λ' becomes the Wilson action of a $U(N)$ lattice gauge theory on a larger lattice. This is not immediately obvious because each site of the new lattice has been labeled by two indices, \mathbf{n}' and \mathbf{m} .

The size of the new lattice depends on whether M and L' have any common divisors. If M and L' are coprime [so that $\gcd(M, L') = 1$], then the new lattice $\Lambda = \mathbb{Z}_L^d$ with $L = ML'$. This reflects the isomorphism $\mathbb{Z}_M \times \mathbb{Z}_{L'} \sim \mathbb{Z}_{ML'}$, as illustrated in figure 4a. Under the isomorphism, the link variable $V_\mu[\mathbf{n}', \mathbf{m}]$ connects $[\mathbf{n}', \mathbf{m}]$ to $[\mathbf{n}' + \mathbf{e}_\mu, \mathbf{m} + \mathbf{e}_\mu]$, which (by definition) is a nearest neighbor on the new lattice.

When M and L' are not coprime, $\mathbb{Z}_M \times \mathbb{Z}_{L'}$ is isomorphic to $\mathbb{Z}_{L'M/Q} \times \mathbb{Z}_Q$, where Q is the greatest common divisor of M and L' . In this case, applying the orbifold projection to the action of the parent theory on lattice Λ' yields Q^d decoupled copies of a daughter theory defined on the lattice $\Lambda = \mathbb{Z}_L^d$ with $L \equiv ML'/Q$. Figure 4b shows an example where $d = 1$, $L' = 6$, and $M = 4$. For this case, the outcome of the orbifold projection is two decoupled copies of the larger volume theory on a 12 site lattice. It is not difficult to show that neutral operators in the parent theory (on lattice $\Lambda' = \mathbb{Z}_{L'}^d$) map to neutral operators in the daughter theory (on lattice $\Lambda = \mathbb{Z}_{ML'/Q}^d$).

A $(\mathbb{Z}_M)^d$ projection does not produce volume enlargement by a factor of M^d unless $\gcd(M, L') = 1$. However, by suitably choosing the projection one can expand the volume by

any desired integer factor. To expand the volume $(L')^d$ by a factor of K^d , one starts with a gauge theory whose number of colors is $N_c = M^d N$, and projects by $(\mathbb{Z}_M)^d$, where M is chosen to satisfy

$$L'M / \gcd(L', M) = L'K. \quad (\text{A.12})$$

This equation for M always has at least one solution, $M = L'K$. The projection yields a daughter theory on a lattice of the desired size $(L'K)^d$ [or rather $\gcd(L', M)^d$ copies thereof].

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